THE COLLAPSE OF BUBBLES IN A VISCOUS LIQUID

(ZAPOLNENIE PUZYR'KOV V VIAZKOI ZHIDKOSTI)

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This paper is concerned with the filling of an empty spherical bubble in a viscous liquid. The existence of two different types of motion is discovered: bubbles, which are smaller than a critical size, are filled slowly in an infinitely long time; the filling of large bubbles takes place rapidly with an unlimited accumulation of energy during collapse.

A quantitative formula is obtained for the critical radius of the bubble.

Assume that for some reason a bubble has formed inside the liquid, which in the future can again be filled under the action of the surrounding pressure. The problem of the filling of a spherical bubble in an inviscid incompressible liquid was studied by Rayleigh. He found, in particular, that the velocity of motion of its surface towards the center grows indefinitely as $r^{-3/2}$, as the bubble completes the process of collapsing, i.e. an unlimited accumulation of energy takes place. This phenomenon is assumed to be a possible explanation for the rapid wear in screw propellers and turbines which operate under cavitation conditions. The collapse of bubbles at a metal surface can damage it severely.

Now let us study the Rayleigh problem for the viscous liquid. Such a statement of the problem corresponds more nearly to the actual conditions, although it still does not give an accurate description of the phenomenon, since it does not take into account the compressibility of the liquid, the inevitable presence of its vapor in the bubble and the possible instability of the slope of the bubble.

Assume that a spherical bubble of radius a was formed in a liquid of density ρ, pressure p_0 (far from the bubble) and viscosity η. There is no pressure inside the bubble and the initial velocity is zero.

The motion of the bubble will be spherically symmetric and can be described by the Navier-Stokes equations of the form

$$\frac{\partial u}{\partial r} + \frac{2u}{r} = 0, \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \tag{1}$$

Here $u(r, t)$ is the velocity, $p(r, t)$ is the pressure.

The viscosity does not enter Equation (1) because in the general équations it occurs in the term

$$\eta\,(\mathrm{grad\ div\ } \mathbf{u} - \mathrm{rot\ rot\ } \mathbf{u})$$

which in our case is identically equal to zero. This is so because $\mathrm{div\ } \mathbf{u} = 0$, since the liquid is incompressible and $\mathrm{rot\ } \mathbf{u} = 0$ because of spherical symmetry.

The normal stress σ_{rr} at the free surface of the bubble is zero (boundary with a vacuum), and since $\sigma_{rr} = -p + 2\eta\, du/dr$ then

$$p_1 = 2\eta\left(\frac{\partial u}{\partial r}\right)_1$$

i.e. the viscosity has entered the boundary condition. Here and in the following, the index 1 denotes a value at the boundary. The second boundary condition is

$$p = p_0 \qquad \text{for } r = \infty$$

From (1) we obtain

$$u\,(r, t) = \frac{q\,(t)}{r^2} \qquad (q\,(t) = u_1 r_1^2)$$

This is substituted into the second equation of (1), integrated from r_1 to ∞, taking into account the boundary conditions on p, and after some simple calculations we obtain

$$\frac{du_1}{dr_1} + \frac{3}{2}\frac{u_1}{r_1} + \frac{p_0}{\rho r_1 u_1} + \frac{4\nu}{r_1^2} = 0 \qquad \left(\nu = \frac{\eta}{\rho}\right) \tag{2}$$

Here ν is the kinematic viscosity. Let us introduce non-dimensional variables

$$\zeta = \frac{u_1}{\sqrt{p_0/\rho}}, \qquad \alpha = \frac{r_1 R}{a} \qquad \left(R = \frac{a}{\nu}\sqrt{\frac{p_0}{\rho}}\right)$$

where R is the Reynolds number. Then Equation (2) becomes

$$\frac{d\zeta}{d\alpha} + \frac{3}{2}\frac{\zeta}{\alpha} + \frac{1}{\zeta\alpha} + \frac{4}{\alpha^2} = 0 \tag{3}$$

The initial condition is of the form $\zeta(R) = 0$ (i.e. $u_1(a) = 0$). As we see, the problem under study has a family of solutions which depends on the number R. One of them, $R = \infty(\nu = 0)$, coincides with the Rayleigh solution. For this family, the velocity near the center grows as $\zeta \sim a^{-3/2}$.

Let us study the behavior of the velocity near the center in the case of nonzero viscosity. For studying the value ζ^{-1} we represent Equation (3) in the form

$$\frac{d\zeta^{-1}}{d\alpha} = \frac{\zeta^{-2}}{\alpha}\left(\frac{3}{2}\zeta + \zeta^{-1} + \frac{4}{\alpha}\right) \tag{4}$$

The point ($a = 0$, $\zeta^{-1} = 0$) for this equation is a multiple singularity, shown in Fig. 1.

The loci of the zeros and infinities are shown in Fig. 1 by means of heavy lines and identified by 0 and ∞, whereas the integral curves are denoted by the light lines. The dot-dash line OA is the only integral curve which emanates from the origin with a finite slope. Its slope x is determined from (4) by substitution of the solution $\zeta^{-1} = x\alpha$ near $a = 0$, and it becomes $x = 1/8$. The locus OB of the zeros passes by lower; its slope is $-3/8$. The integral curves above OA enter a node, where $\zeta \sim a^{-3/2}$. Below OA they represent a saddle where

$$\zeta \to -{}^1\!/_4\, \alpha, \quad \text{or} \quad u_1 \to -\frac{r_1 p_0}{4\eta}$$

as $a \to 0$. Both of these results are verified by substitution in (4) with $a \to 0$. The curve which corresponds to the solution is that which leads towards a point corresponding to the initial condition, i.e. $a = R$, $\zeta = 0$.

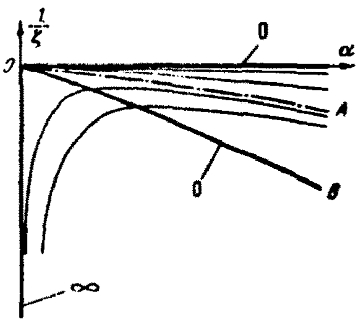

Fig. 1.

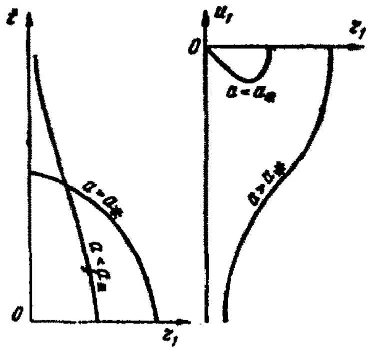

Fig. 2.

Generally speaking, for different numbers R the solutions may belong to different families: in the case of the node they correspond to an unlimited increase of the velocity $\zeta \sim a^{-3/2}$; in the case of the saddle point they correspond to a decelerated motion of the bubble with $\zeta \sim a$, as a result of which its filling occurs only after an infinite time.

In the intermediate case, which corresponds to the dividing line *OA*, the filling occurs after a finite time, and the velocity near the center grows as $\zeta \sim a^{-1}$. The Reynolds number corresponding to the dividing line is the critical Reynolds Number. It separates two fundamentally different classes of solutions. Its critical value $R = R_*$ can be determined by constructing the separatrix from $a = 0$ (near that point its asymptotic representation is known: $\zeta = -1/8\, a^{-1}$) to $\zeta = 0$, where $a = R$. Its construction, which is accomplished with the aid of numerical integration (3), yields

$$R_* = 8.4 \qquad \left(R = \frac{a}{\nu}\sqrt{\frac{p_0}{\rho}}\right)$$

For $R < 8.4$ the filling of the bubble takes place slowly in an infinitely long time. The accumulation of energy is completely dissipated by the viscosity.

For $R > 8.4$ the velocity near the focus grows indefinitely as in the Rayleigh problem (without viscosity), i.e. as const $r_1^{-3/2}$, but with a smaller value of $|\text{const}|$. In the interim case for $R_* = 8.4$ the bubble is filled in finite time, and the velocity of the focus grows indefinitely, but more weakly, as r_1^{-1}.

The schemes of all three cases of motion are shown in Fig. 2. For given p_0, ρ and ν one can speak of a critical radius a_* of the bubble

$$a_* = 8.4\,\nu\sqrt{\frac{\rho}{p_0}}$$

For $a < a_*$ the energy accumulation is completely dissipated by the viscosity.

The velocity at the end of the filling of small bubbles

$$u_1 = -\frac{r_1 p_0}{4\eta}$$

does not depend on the density and on their initial radius a. In practice, the critical radius a_* is very small, i.e. the viscosity dissipates the accumulation only in bubbles of very small size; for instance, for $p_0 = 1$ atm $= 10^6$ bar in water ($\nu = 0.01$, $\rho = 1$) $a_* = 0.8$ micron, and in glycerin ($\nu = 6.8$, $\rho = 0.8$) $a_* = 0.5$ mm.

Translated by M.I.Y.